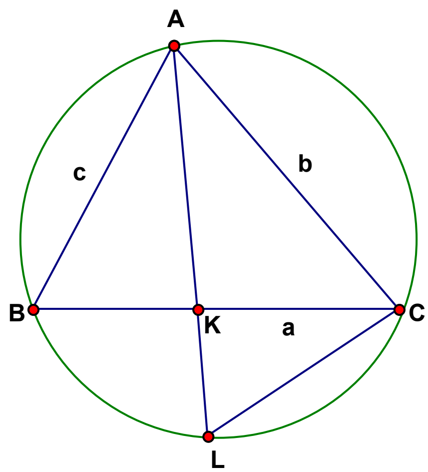
One Inequality with angle bisectors.
Question 3.
Let ABC be a triangle inscribed in a circle and let $l_a = \dfrac{m_a}{M_a}, l_b = \dfrac{m_b}{M_b}, l_c = \dfrac{m_c}{M_c}$,
where m_a, m_b, m_c are the lengths of the angle bisectors and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle.
Prove that $\dfrac{l_a}{\sin^2 A} + \dfrac{l_b}{\sin^2 B} + \dfrac{l_c}{\sin^2 C} \geq 3$,
and that equality holds iff ABC is equilateral triangle.
Solution by Arkady Alt, San Jose, California, USA.



Let K and L be, intersection points of bisector of angle A with BC and circumcircle, respectively and let F be area of $\triangle ABC$. Similarity of triangles ABC and ALC implies
$\dfrac{AB}{AK} = \dfrac{AL}{AC} \Leftrightarrow \dfrac{c}{m_a} = \dfrac{M_a}{b} \Leftrightarrow M_a m_a = bc.$
Hence, $\dfrac{l_a}{\sin^2 A} = \dfrac{m_a}{M_a \sin^2 A} = \dfrac{m_a^2}{M_a m_a \sin^2 A} = \dfrac{m_a^2}{bc \sin^2 A} = \dfrac{m_a^2 bc}{b^2 c^2 \sin^2 A} = \dfrac{m_a^2 bc}{4F^2}$ and,
therefore, $\sum \dfrac{l_a}{\sin^2 A} \geq 3 \Leftrightarrow \sum \dfrac{m_a^2 bc}{4F^2} \geq 3 \Leftrightarrow \sum m_a^2 bc \geq 12F^2.$
Since $m_a^2 = bc - \dfrac{a^2 bc}{(b+c)^2} \geq bc - \dfrac{a^2}{4(b+c)^2}$ (because $\dfrac{bc}{(b+c)^2} \leq \dfrac{1}{4} \Leftrightarrow (b-c)^2 \geq 0$)
then $\sum m_a^2 bc \geq \sum \left(b^2 c^2 - \dfrac{a^2 bc}{4}\right)$. Thus, remains to prove inequality
$\sum \left(b^2 c^2 - \dfrac{a^2 bc}{4}\right) \geq 12F^2 \Leftrightarrow \sum (4b^2 c^2 - a^2 bc) \geq 3 \cdot 16F^2 \Leftrightarrow$
$\sum (4b^2 c^2 - a^2 bc) \geq 3 \sum (2b^2 c^2 - a^4) \Leftrightarrow 3(a^4 + b^4 + c^4) \geq 2(a^2 b^2 + b^2 c^2 + c^2 a^2) + abc(a + b +$
Noting that $a^4 + b^4 + c^4 \geq a^2 b^2 + b^2 c^2 + c^2 a^2$, $a^4 + b^4 + c^4 \geq \dfrac{(a+b+c)(a^3+b^3+c^3)}{3}$
(by Chebishev's Inequality) and $a^3 + b^3 + c^3 \geq 3abc$ (by AM-GM Inequality)
we finally obtain $3(a^4 + b^4 + c^4) = 2(a^4 + b^4 + c^4) + (a^4 + b^4 + c^4) \geq$
$2(a^2 b^2 + b^2 c^2 + c^2 a^2) + abc(a + b + c).$